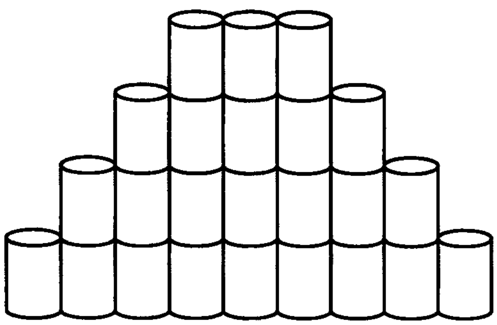
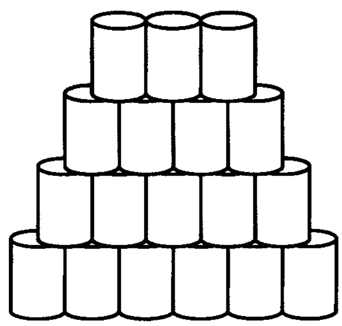
Chapter 4.

Series.

Consider the pile of cans shown below left.
The top row contains 3 cans, the second row 5 cans, the third row 7 cans and so on.

A more stable arrangement could be the arrangement shown below right.
In this case the top row contains 3 cans, the second row 4 cans, the third row 5 cans and so on.





For both situations the number of cans in each row progress arithmetically. Numbering the rows from the top we have

Arithmetic sequence	Arithmetic sequence
First term 3	First term 3
Common difference 2	Common difference 1

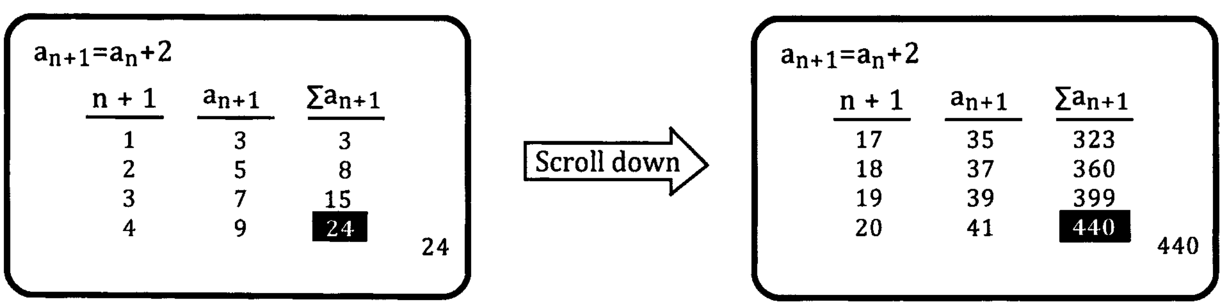
Thus in each case we could determine the number of cans in the 20th row:

$$T_{20} = 3 + 19 \times 2 = 41 \qquad\qquad T_{20} = 3 + 19 \times 1 = 22$$

However if we were going to attempt to build these 20 row piles of cans we would also want to know how many cans each design would require altogether. I.e. we would want to evaluate

$$3 + 5 + 7 + 9 + 11 + 13 + \ldots + 41 \qquad \text{and} \qquad 3 + 4 + 5 + 6 + 7 + 8 + \ldots + 22$$

As we saw in the previous chapter, we could define each sequence recursively and use a calculator with the ability to show both the terms and partial sums of such a sequence, as shown below for the arithmetic sequence with first term 3 and common difference 2.



When we sum the terms of a sequence we produce a **series**.

Thus	3,	5,	7,	9,	11,	13,	15,	is a sequence
	3 +	5 +	7 +	9 +	11 +	13 +	15	is the corresponding series

Arithmetic series.

An arithmetic series has the form

$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + (a+5d) + \ldots.$$

Using S_n for the sum of the first n terms

$$S_1 = T_1 = a$$
$$S_2 = T_1 + T_2 = a + (a+d)$$
$$S_3 = T_1 + T_2 + T_3 = a + (a+d) + (a+2d)$$
$$S_4 = T_1 + T_2 + T_3 + T_4 = a + (a+d) + (a+2d) + (a+3d)$$
$$S_5 = T_1 + T_2 + T_3 + T_4 + T_5 = a + (a+d) + (a+2d) + (a+3d) + (a+4d)$$

$$S_n = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + \ldots + T_n$$
$$= a + (a+d) + (a+2d) + (a+3d) + (a+4d) + (a+5d) + \ldots + a + (n-1)d$$

Example 1

For the arithmetic sequence 3, 7, 11, 15, 19, 23, ... determine
(a) S_2,
(b) S_5,
(c) S_7.

(a) $S_2 = T_1 + T_2$
$= 3 + 7$
$= 10$

(b) $S_5 = T_1 + T_2 + T_3 + T_4 + T_5$
$= 3 + 7 + 11 + 15 + 19$
$= 55$

(c) $S_7 = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$
$= 3 + 7 + 11 + 15 + 19 + 23 + 27$
$= 105$

Example 2

An arithmetic sequence has an nth term given by $T_n = 3n - 2$. Determine the first five terms of this sequence and hence determine S_1, S_2, S_3, S_4 and S_5, the first five terms of the corresponding series.

$T_n = 3n - 2$ $\therefore$

$T_1 = 3(1) - 2 = 1$ and $S_1 = 1$
$T_2 = 3(2) - 2 = 4$ and $S_2 = 1 + 4 = 5$
$T_3 = 3(3) - 2 = 7$ and $S_3 = 1 + 4 + 7 = 12$
$T_4 = 3(4) - 2 = 10$ and $S_4 = 1 + 4 + 7 + 10 = 22$
$T_5 = 3(5) - 2 = 13$ and $S_5 = 1 + 4 + 7 + 10 + 13 = 35$

Alternatively questions like the previous two examples could be solved using a spreadsheet or using the ability of some calculators to display the terms and partial sums of sequences. However, whilst use of such technology can be very useful at times do note that for the previous two examples the "pencil and paper" methods demonstrated were really very straightforward.

A formula for S_n of an arithmetic series.

$$S_n = T_1 + T_2 + T_3 + \dots + T_{n-2} + T_{n-1} + T_n$$

Thus for the general arithmetic series with first term a and common difference d:

$$S_n = a + a+d + a+2d + \dots + a+(n-3)d + a+(n-2)d + a+(n-1)d$$

If we write these terms in the reverse order it also follows that:

$$S_n = a+(n-1)d + a+(n-2)d + a+(n-3)d + \dots + a+2d + a+d + a$$

Adding these two versions of S_n gives:

$$2S_n = \underbrace{2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d + \dots + 2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d}_{n \text{ lots of } 2a+(n-1)d}$$

$$= n[\,2a + (n-1)d\,]$$

Thus for an arithmetic progression with first term a and common difference d:

$$\boxed{S_n = \frac{n}{2}[\,2a + (n-1)\,d\,]}$$

If we use ℓ for the last term, instead of $a + (n-1)d$, this rule can be written as

$$\boxed{S_n = \frac{n}{2}[\,a + \ell\,]}$$

Example 3

For the arithmetic series 10 + 17 + 24 + 31 + 38 + ... determine (a) S_2, (b) S_{50}.

(a) $S_2 = 10 + 17$

$= 27$

(b) Using $S_n = \frac{n}{2}[\,2a + (n-1)\,d\,]$ with n = 50, a = 10 and d = 7 gives

$S_{50} = \frac{50}{2}[\,2(10) + 49\,(7)\,]$

$= 9075$

Make sure you can obtain these same answers using

- a calculator capable of displaying the terms and sums of a sequence.
- a spreadsheet.

Example 4

For the sequence 9, 13, 17, 21, 25, 29

(a) Show that 157 is the 38th term.

(b) Evaluate 9 + 13 + 17 + 21 + 25 + 29 + ... 157.

(a) For an arithmetic progression with first term a and common difference d,

$$T_n = a + (n-1)d$$

Thus for the given AP: $T_{38} = 9 + 37(4)$

$= 157$ as required

(b) Using $S_n = \frac{n}{2}[a + \ell]$ with n = 38, a = 9 and ℓ = 157

$$S_{38} = \frac{38}{2}[9 + 157]$$

$$= 3154$$

Thus 9 + 13 + 17 + 21 + 25 + 29 + ... 157 = 3154

Example 5

A fifteen year service contract involves a company paying \$12000 in the first year of a contract with an annual increase of \$800 every year after that for the life of the contract. How much will the company have paid on this contract in total by the end of the fifteen years?

We require $S_{15} = \$12\,000 + \$12\,800 + \$13\,600 + \ldots + T_{15}$.

Using $S_n = \frac{n}{2}[2a + (n-1)d]$ with n = 15, a = 12000 and d = 800 gives

$$S_{15} = \frac{15}{2}[2(12\,000) + 14(800)]$$

$$= \$264\,000$$

The company will have paid \$264000 under this contract by the end of the fifteen years.

Example 6

A company borrows \$24000. They agree that at then end of each month the remaining balance is calculated and the company pays 1% of this remaining balance as interest payments, and then \$1000 to reduce the remaining balance. In this way the loan will be repaid in two years. How much will the loan cost the company in interest payments?

End of month	1	2	3	4	5	...	24
Remaining balance	\$24000	\$23000	\$22000	\$21000	\$20000	...	\$1000
1% of remaining balance	\$240	\$230	\$220	\$210	\$200	...	\$10
Total of interest payments	\$240 +	\$230 +	\$220 +	\$210 +	\$200 +	...	+ \$10

$$= \frac{24}{2}[\$240 + \$10]$$

$$= \$3000$$

The company pays \$3000 in interest repayments.

Note: These "real life questions" need care. It is easy to confuse a year number with a term number and they may not always match. It is wise to list the first few terms and to think carefully how many terms are required, what is the first term, what is the common difference etc.

Exercise 4A

1. For the arithmetic sequence 8, 14, 20, 26, 32, ... determine (a) S_4, (b) S_5, (c) S_6.

2. For the arithmetic sequence 28, 25, 22, 19, 16, ... determine (a) S_2, (b) S_6, (c) S_1.

3. For the arithmetic sequence −6, −3, 0, 3, 6, ... determine (a) S_2, (b) S_5, (c) S_6.

4. An arithmetic sequence has an nth term given by $T_n = 5n + 1$. Determine the first four terms of this sequence and hence determine S_1, S_2, S_3 and S_4, the first four terms of the corresponding series.

5. An arithmetic sequence is defined by the recursive rule:
$$T_{n+1} = T_n + 3, \quad T_1 = 11.$$
Determine the first four terms of this sequence and hence determine S_1, S_2, S_3 and S_4, the first four terms of the corresponding series.

6. An arithmetic sequence has an nth term given by $T_n = 25 - 3n$. Determine the first four terms of this sequence and hence determine S_1, S_2, S_3 and S_4, the first four terms of the corresponding series.

7. Determine the first 5 terms of a sequence given that the corresponding series is such that: $S_1 = 25$, $S_2 = 57$, $S_3 = 96$, $S_4 = 142$, $S_5 = 195$.
Is the sequence arithmetic?

8. Determine the first 5 terms of a sequence given that the corresponding series is such that: $S_1 = 1$, $S_2 = 5$, $S_3 = 14$, $S_4 = 30$, $S_5 = 55$.
Is the sequence arithmetic?

9. For the arithmetic series 5 + 16 + 27 + 38 + 49 + 60...determine (a) S_3, (b) S_{40}.

10. For the arithmetic series 60 + 58 + 56 + 54 + 52 + 50...determine (a) S_3, (b) S_{60}.

11. Find the sum of the first 100 counting numbers: 1, 2, 3, 4, 5, ..., 100.

12. For the sequence 16, 20, 24, 28, 32, 36, ...
    (a) Show that the 29th term is 128.
    (b) Evaluate 16 + 20 + 24 + 28 + 32 + 36 + ... + 128.

13. For the sequence 9, 26, 43, 60, 77, 94, ...
    (a) Show that the 41st term is 689.
    (b) Evaluate 9 + 26 + 43 + 60 + 77 + 94 + ... + 689.

14. Part of a cyclist's training program involves him in riding 20 km on the first day of the month, 22 km on the second day, 24 on the third and so on, the distances increasing in arithmetic progression.
    How far will he cycle on the 30th day of the month?
    How far will he cycle in total during these 30 days?

15. A farmer plants trees in a corner of one of his paddocks. He plants them in rows with the first row containing 5 trees, the second containing 7 trees, the third containing 9 trees and so on
    How many trees will he need to plant if he is to plant 15 rows?

16. A twelve month equipment hire contract involves a company paying $4000 at the end of the first month, $3750 at the end of the second month, and so on, with each monthly payment decreasing by $250 after that, for the life of the contract. How much will the company have paid in total on this contract by the time they make the last payment at the end of the twelfth month?

17. Jack is offered two jobs , one with company A and the other with company B.
    Company A offers him $65 000 in the first year, increasing by $2500 in each subsequent year.
    Company B offers him $68 000 in the first year, increasing by $1200 in each subsequent year.
    How much would he receive from each company if he were to work for them for ten years?

18. A company borrows $36 000. They agree that at the end of each month the remaining balance is calculated and the company pays 2% of this remaining balance, as interest payments, and then pays $2000 to reduce the remaining balance. In this way they will repay the loan in eighteen months.
    How much will the loan cost the company in interest payments?

Geometric series.

Example 7

A geometric sequence has an n^{th} term given by $T_n = 3(2)^n$. Determine the first five terms of this sequence and hence determine S_1, S_2, S_3, S_4, and S_5, the first five terms of the corresponding geometric series.

$$T_n = 3(2)^n \quad \therefore \quad T_1 = 3(2)^1 = 6 \quad \text{and} \quad S_1 = 6$$
$$T_2 = 3(2)^2 = 12 \quad \text{and} \quad S_2 = 6 + 12 = 18$$
$$T_3 = 3(2)^3 = 24 \quad \text{and} \quad S_3 = 6 + 12 + 24 = 42$$
$$T_4 = 3(2)^4 = 48 \quad \text{and} \quad S_4 = 6 + 12 + 24 + 48 = 90$$
$$T_5 = 3(2)^5 = 96 \quad \text{and} \quad S_5 = 6 + 12 + 24 + 48 + 96 = 186$$

A formula for S_n of a geometric series.

For the general geometric series with first term a and common ratio r:

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} \qquad [1]$$

Multiply by r:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n \qquad [2]$$

Equation [1] – equation [2]:

$$S_n - rS_n = a + 0 + 0 + 0 + \dots\dots + 0 - ar^n$$
$$S_n(1 - r) = a - ar^n$$

For the geometric sequence with first term a and common ratio *r*

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad r \neq 1$$

If *r* is not between –1 and 1 this formula is easier to use in the form

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad r \neq 1$$

Example 8

Evaluate S_{10} for the series 512 + 768 + 1152 + 1728 + ...

The given sequence of terms are geometric with first term 512 and common ratio 1·5.

Using
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{512(1{\cdot}5^{10} - 1)}{1{\cdot}5 - 1}$$
$$= 58025$$

Example 9

A geometric progression has first term of 40 and a common ratio of 0·75.
Find S_{12}, the sum of the first twelve terms, giving your answer correct to two decimal places.

In this case r is between −1 and 1 so we use $S_n = \frac{a(1 - r^n)}{1 - r}$

$$S_{12} = \frac{40(1 - 0{\cdot}75^{12})}{1 - 0{\cdot}75}$$

$$= 154{\cdot}93 \quad \text{correct to 2 d.p.}$$

Example 10

What is the least number of terms required for the geometric series

$$10 + 15 + 22{\cdot}5 + 33{\cdot}75 + \ldots$$

to have a sum that exceeds 1 000 000?

In this case r = 1·5 and a = 10. Use $S_n = \frac{a(r^n - 1)}{r - 1}$ and let $S_n = 1\,000\,000$

i.e. $$1\,000\,000 = \frac{10(1{\cdot}5^n - 1)}{1{\cdot}5 - 1}$$

$$50\,000 = 1{\cdot}5^n - 1$$

By trial and adjustment n = bigger than 26 but less than 27.
However n must be a positive integer. Thus at least 27 terms are needed for the given sum to exceed 1 000 000.

(Alternatively $50\,000 = 1{\cdot}5^n - 1$ can be solved using a calculator with a "solve facility".)

Example 11

On 1st January 2014 Mrs Smith starts a savings plan by investing \$500 into an account that guarantees 5% interest per annum provided she commits to invest a further \$500 on the 1st January every year. The plan will finish on 31st December 2023. How much should this savings plan be worth on 31st December 2023 (to the nearest 10 cents)?

Date	Value of investment
1st Jan 2014	\$500
1st Jan 2015	\$500 + \$500 × 1·05
1st Jan 2016	\$500 + \$500 × 1·05 + \$500 × $1{\cdot}05^2$
1st Jan 2017	\$500 + \$500 × 1·05 + \$500 × $1{\cdot}05^2$ + \$500 × $1{\cdot}05^3$
	⋮
1st Jan 2023	\$500 + \$500 × 1·05 + \$500 × $1{\cdot}05^2$ + ... + \$500 × $1{\cdot}05^9$
31st Dec 2023	\$500 × 1·05 + \$500 × $1{\cdot}05^2$ + ... + \$500 × $1{\cdot}05^9$ + \$500 × $1{\cdot}05^{10}$

Thus the final value is S_{10} for a geometric series with 1st term \$500 × 1·05 and r = 1·05.

$$S_{10} = \frac{\$525(1{\cdot}05^{10} - 1)}{1{\cdot}05 - 1}$$

$$= \$6603{\cdot}40 \quad \text{to the nearest 10 cents.}$$

Alternatively:

- The previous example could be solved using a recursive definition, as in the previous chapter. With $T_1 = 500$ and $T_{n+1} = T_n \times 1{\cdot}05 + 500$, the required answer would be given by $T_{11} - 500$. (Or with $T_0 = 500$ and $T_{n+1} = T_n \times 1{\cdot}05 + 500$, and the required answer given by $T_{10} - 500$.)
- Some calculators have built in financial programs that will calculate values of accounts in situations like that of the last example. Whilst you are encouraged to explore the capability of your calculator in this regard make sure that if required you can calculate values using geometric series and recursively defined sequences.

Exercise 4B

1. A geometric sequence has an n^{th} term given by $T_n = 2(3)^n$. Determine the first four terms of this sequence and hence determine S_1, S_2, S_3, and S_4, the first four terms of the corresponding series.

2. A geometric sequence is defined by the recursive rule $T_{n+1} = 1{\cdot}5\, T_n$, $T_2 = 24$. Determine the first four terms of this sequence and hence determine S_1, S_2, S_3 and S_4, the first four terms of the corresponding geometric series.

3. Determine the first 5 terms of a sequence given that the corresponding series is such that: $S_1 = 1$, $S_2 = 2$, $S_3 = 4$, $S_4 = 7$, $S_5 = 12$.
   Is the sequence geometric?

4. Determine the first 5 terms of a sequence given that the corresponding series is such that: $S_1 = 8$, $S_2 = 32$, $S_3 = 104$, $S_4 = 320$, $S_5 = 968$.
   Is the sequence geometric?

5. Evaluate S_{15} for the series $1 + 2 + 4 + 8 + 16 + \ldots$.

6. Evaluate S_{11} for the series $20\,480 + 10\,240 + 5\,120 + 2\,560 + 1\,280 + \ldots$.

7. A geometric sequence has a first term of 256 and a common ratio of 2·5.
   Find S_9, the sum of the first nine terms.

8. A geometric sequence has a first term of 62 500 and a common ratio of 0·4.
   Find S_9, the sum of the first nine terms.

9. In the spreadsheet shown on the right, column B shows the first 7 terms of a geometric sequence. If S_n represents the sum of the first n terms determine S_6, S_7 and S_8.

	A	B
1	Term	Value
2	1	2.25
3	2	9
4	3	36
5	4	144
6	5	576
7	6	2304
8	7	9216

10. What is the least number of terms required for the geometric series

$$5 + 10 + 20 + 40$$

to have a sum that exceeds 5 000 000?

11. What is the least number of terms required for the geometric series

$$28 + 42 + 63 + 94{\cdot}5$$

to have a sum that exceeds 1 000 000?

12. A GP has a third term of 24 and a fourth term of 96. Find T_{10} and S_{10}.

13. A sports star negotiates a contract with a sports equipment company that ensures she will be paid \$50 000 in the first year, \$57 500 in the second year, \$66 125 in the third year and so on, the annual amounts continuing in this geometric progression. Calculate the total amount the star will receive in the 10 years this contract is for. (Give your answer to the nearest \$1 000.)

14. In its first year of operation a mine yielded 5000 tonnes of a particular mineral. In the second year the yield was approximately 110% of the first year's figures. In each subsequent year the yield continued to be approximately 110% of the previous year's yield.
What tonnage of the mineral did the mine yield in
(a) its second year of operation,
(b) its third year of operation,
(c) its fourth year of operation?
(d) What total tonnage of the mineral did the mine yield in its first twelve years of operation?

15. A company's profit in its first year of operation was \$60 000. Each year thereafter the annual profit increased by approximately 15% of the previous year's profit. Calculate the company's profit in (a) the second year, (b) the third year, (c) the tenth year.
Find the total profit the company makes in its first ten years of operation.

16. \$1200 is deposited on 1st January 2014 to open an account. On each subsequent anniversary of this date a further \$1200 is deposited into the account. The account earns interest at the rate of 10% per annum, compounded annually.
Copy and complete the 1/1/18 line in the following table for this situation.

Date	Value of deposit made on					Total value
	1/1/14	1/1/15	1/1/16	1/1/17	1/1/18	
1/1/14	\$1200					\$1200
1/1/15	$\$1200 \times 1{\cdot}1$	\$1200				\$2520
1/1/16	$\$1200 \times 1{\cdot}1^2$	$\$1200 \times 1{\cdot}1$	\$1200			\$3972
1/1/17	$\$1200 \times 1{\cdot}1^3$	$\$1200 \times 1{\cdot}1^2$	$\$1200 \times 1{\cdot}1$	\$1200		\$5569·20
1/1/18						

How much will be in the account immediately following the deposit of \$1200 made on the 1st January 2029, to the nearest dollar?

17. A person deposits \$1000 on the 1st January 2015 and a further \$1000 on the 1st January of each subsequent year up to and including 2024.
The investment gains interest at 7% per annum compounded annually. How much will be in the account when it is closed on 31st December 2024? (Give your answer to the nearest dollar.)

18. As a new employee becomes more accustomed to the machine she is operating the number of units she produces per day increases. Her daily production approximates closely to a geometric progression for the first fifteen days with 2500 produced on the first day, 2550 on day two, 2601 on day three etc. The employee maintains the fifteenth day's output thereafter.
    (a) How many units does she produce on day 4?
    (b) How many units does she produce on day 15?
    (c) How many units does she produce on day 16?
    (d) Find the total number of units this employee produces in her first fifteen days working with the company. (To the nearest hundred.)
    (e) Find the total number of units this employee produces in her first forty days working with the company. (To the nearest hundred.)

19. 8000 tonnes of a particular mineral were mined in each of the first three years of a mines operation. In the fourth year the quantity mined was 90% of the third year's output, the fifth year was 90% of the fourth year and so on. The mine was closed at the end of the first year in which the amount mined fell below 1900 tonnes.
    (a) For how many years did the mine remain open?
    (b) What total tonnage of the mineral was mined from the mine?

20. After the birth of their son, Mr and Mrs Jacques decide to open an account for him commencing when he reaches the age of 1. They wish to invest the same fixed amount on each birthday from 1 to 21 such that, immediately following the 21st payment, the account would hold \$50 000. The account earns interest at 9·5% per annum, compounded annually.
Suppose the fixed amount they wish to invest is \$P. The table below shows how the value of the account grows during the first four years.

	Value of deposit made on			
Birthday	1st birthday	2nd birthday	3rd birthday	4th birthday
1	\$P			
2	\$P × 1·095	\$P		
3	\$P × $1{\cdot}095^2$	\$P × 1·095	\$P	
4	\$P × $1{\cdot}095^3$	\$P × $1{\cdot}095^2$	\$P × 1·095	\$P

Thus immediately after the 4th birthday payment the account is worth

$$\$P + \$P \times 1{\cdot}095 + \$P \times 1{\cdot}095^2 + \$P \times 1{\cdot}095^3$$

(a) What is the first term, common ratio and number of terms for the similar expression for the value of the account immediately after the 21st payment has been made?
(b) Determine the value of P correct to 1 decimal place.

Infinite geometric series.

Each of the following tables involve geometric progressions.
Copy and complete the six tables.

1. a = 64, r = 5

n	T_n	S_n
1	64	64
2	320	384
3	1600	1984
4	8000	9984
5		
6		
7		

2. a = 0·4, r = 4

n	T_n	S_n
1	0·4	0·4
2	1·6	2·0
3	6·4	8·4
4	25·6	34·0
5		
6		
7		

3. a = 100, r = 1·8

n	T_n	S_n
1	100	100
2	180	280
3	324	604
4	583·2	1187·2
5		
6		
7		

4. a = 64, r = 0·2

n	T_n	S_n
1	64	64
2	12·8	76·8
3	2·56	79·36
4	0·512	79·872
5		
6		
7		

5. a = 10, r = 0·5

n	T_n	S_n
1	10	10
2	5	15
3	2·5	17·5
4	1·25	18·75
5		
6		
7		

6. a = 90, r = 0·4

n	T_n	S_n
1	90	90
2	36	126
3	14·4	140·4
4	5·76	146·16
5		
6		
7		

Reading down the S_n columns in your completed tables you should notice that in each table the numbers are getting bigger and bigger as we go down the column. However notice that whilst in the first three tables these numbers get very big, in the last three tables the numbers are increasing but by a smaller and smaller amount each time.

Indeed it seems that:

the numbers in the S_n column for table 4 are heading towards 80,
the numbers in the S_n column for table 5 are heading towards 20,
the numbers in the S_n column for table 6 are heading towards 150.

This should be no surprise if we consider the formula: $S_n = \dfrac{a(1-r^n)}{1-r}$.

For tables 4, 5 and 6 the common ratio is between −1 and 1 and all such numbers get very small when raised to a large power. Thus in our formula r^n will get smaller and smaller as n gets bigger and bigger and so, as n gets large $S_n \approx \dfrac{a}{1-r}$.

For table 4, a = 64 and r = 0·2. Thus as n gets large $S_n \approx \dfrac{64}{1-0{\cdot}2}$

$= 80$, as we found.

For table 5, a = 10 and r = 0·5. Thus as n gets large $S_n \approx \dfrac{10}{1-0{\cdot}5}$

$= 20$, as we found.

For table 6, a = 90 and r = 0·4. Thus as n gets large $S_n \approx \dfrac{90}{1-0{\cdot}4}$

$= 150$, as we found.

We call $\dfrac{a}{1-r}$ the formula for the *sum to infinity* of a GP with first term a and common ratio r. This is the value that S_n gets closer and closer to as n gets bigger and bigger.

Remember • This concept only makes sense for geometric progressions for which $-1 < r < 1$.

Using the symbol "∞" to represent infinity:

> For a geometric series $a + ar + ar^2 + ar^3 + \ldots$ with $-1 < r < 1$,
> $$S_\infty = \frac{a}{1-r}$$

Example 12

Determine the sum to infinity of a geometric progression with first term 36 and common ratio 0·25.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{36}{1-0{\cdot}25} \\ &= 48 \end{aligned}$$

The sum to infinity for the given series is 48.

The table below shows T_n and S_n for n = 1 to 10 for the geometric progression of the previous example, i.e. for the GP with first term 36 and common ratio 0·25 for which we found $S_\infty = 48$

n	T_n	S_n
1	36	36
2	9	45
3	2·25	47·25
4	0·5625	47·8125
5	0·140625	47·953125
6	0·03515625	47·98828125
7	0·0087890625	47·9970703125
8	0·002197265625	47·999267578125
9	0·00054931640625	47·99981689453125
10	0·0001373291015625	47·9999542236328125

Example 13
For each of the following geometric series determine whether S_∞ exists and, if it does, determine its value.
(a) 120 + 90 + 67·5 + ...
(b) 64 + 96 + 144 + ...

(a)
$$r = \frac{90}{120} = 0{\cdot}75$$
$$S_\infty = \frac{120}{1 - 0{\cdot}75} = 480$$

(b)
$$r = \frac{96}{64} = 1{\cdot}5$$
r is not between −1 and 1.

S_∞ does not exist.

Example 14
A patient's body absorbs a certain drug in such a way that whatever is in the body at a particular time, 50% remains in the body 24 hours later.
Every 24 hours for the rest of his life the patient has to give himself an injection containing 20 mg of the drug. In the long term how many mg of the drug will be in the patient's body
(a) immediately after each injection,
(b) immediately prior to each injection?

Number of mg in patient's body immediately before and immediately after injection.		
	Immediately before	Immediately after
1st injection	0	20
2nd injection	0·5 (20) = 10	10 + 20
3rd injection	0·5 (10 + 20) = 5 + 10	5 + 10 + 20
4th injection	2·5 + 5 + 10	2·5 + 5 + 10 + 20

Thus in the long term the amount in the patients body

(a) immediately after each injection $= 20 + 10 + 5 + 2{\cdot}5 + \ldots$

$$= \frac{20}{1 - 0{\cdot}5}$$

$= 40$ mg

(b) immediately before each injection $= 10 + 5 + 2{\cdot}5 + \ldots$

$$= \frac{10}{1 - 0{\cdot}5}$$

$= 20$ mg

Exercise 4C

1. The table below shows T_n and S_n, from n = 1 to n = 8, for 3 GPs, A, B and C.
   For each of the progressions
   (a) determine the common ratio,
   (b) state whether S_∞ exists and, if it does, state its value.

	Geometric progression A		Geometric progression B		Geometric progression C	
n	T_n	S_n	T_n	S_n	T_n	S_n
1	24	24	8	8	35	35
2	9·6	33·6	12	20	10·5	45·5
3	3·84	37·44	18	38	3·15	48·65
4	1·536	38·976	27	65	0·945	49·595
5	0·6144	39·5904	40·5	105·5	0·2835	49·8785
6	0·24576	39·83616	60·75	166·25	0·08505	49·96355
7	0·098304	39·934464	91·125	257·375	0·025515	49·989065
8	0·0393216	39·9737856	136·6875	394·0625	0·0076545	49·9967195

2. For each of the following geometric series determine whether S_∞ exists and, if it does, determine its value.
   (a) $100 + 50 + 25 + \ldots$
   (b) $100 + 75 + 56{\cdot}25 + \ldots$
   (c) $100 + 110 + 121 + \ldots$
   (d) $90 + 72 + 57{\cdot}6 + \ldots$
   (e) $56 + 70 + 87{\cdot}5 + \ldots$
   (f) $90 - 72 + 57{\cdot}6 - \ldots$
   (g) $0{\cdot}6 + 0{\cdot}2 + 0{\cdot}0\overline{6} + \ldots$
   (h) $2304 - 288 + 36 - \ldots$

3. A geometric series with a first term of 48 has a sum to infinity of 120.
   Determine the common ratio of this series.

4. A geometric series with a common ratio of 0·45 has a sum to infinity of 120.
   Determine the first term of this series.

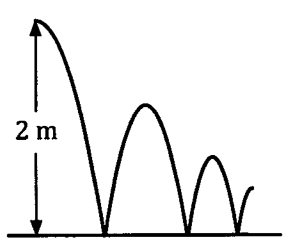
5. A patient's body absorbs a certain drug in such a way that whatever is in the body at a particular time, 40% remains in the body 24 hours later.
Every 24 hours for the rest of his life the patient has to give himself an injection containing 15 mg of the drug.
Copy and complete the following table:

Number of mg in patient's body immediately before and after injection.		
	Immediately before	Immediately after
1st injection	0	15
2nd injection		
3rd injection		
4th injection		
5th injection		

In the long term how many mg of the drug will be in the patient's body
(a) immediately after each injection,
(b) immediately prior to each injection?

6. An athlete is taking part in a test to assess his endurance and fitness.
The athlete manages to complete a particular exercise 50 times in the first minute, 40 times in the second minute and 32 times in the third minute without stopping for a rest. If the athlete were to continue this activity, and if the number of times he completed the particular exercise in each minute continued the geometric progression of the first three minutes, theoretically what is the greatest number of times he could complete the exercise without stopping for a rest?
Discuss the reality of this theoretical greatest number.

7. When a particular ball is dropped onto a horizontal surface the height it reaches on its first bounce is 60% of the height of the previous bounce. Subsequent bounce heights are 60% of the height of the previous bounce. If the ball is dropped from a height of 2 metres, onto a horizontal surface, find



(a) the height reached on the first bounce,
(b) the height reached on the sixth bounce,
(c) the total vertical distance the ball travels until the bouncing ceases.

8. When a particular ball is dropped onto a horizontal surface the height it reaches on its first bounce is 40% of the height of the previous bounce. Subsequent bounce heights are 40% of the height of the previous bounce. If the ball is dropped from a height of 5 metres, onto a horizontal surface, find the total vertical distance the ball travels before coming to rest (to the nearest centimetre).

Miscellaneous Exercise Four.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Express each of the following as a power of 2.

(a) 64 (b) 256 (c) $2^3 \times 2 \times 2 \times 2 \times 2$

(d) $2 \times 2 \times 2 \times 2 \times 2 \div 2^3$ (e) $2^6 \times 2^4$ (f) $2^6 \div 2^4$

(g) $4 \times 8 \times 16 \times 32$ (h) 1 (i) $6^2 - 2^2$

Evaluate each of the following without the use of a calculator.

2. 2^{-1} 3. $4^8 \div 4^6$ 4. $\left(\frac{3}{2}\right)^2$

5. 18^0 6. $(4^{0.5})^6$ 7. $5^6 \times 5^{-8}$

8. $\dfrac{3^7 \times 27^2}{3^{14}}$ 9. $\dfrac{5^8 \div 5^4}{125}$ 10. $\dfrac{7^{10} \div 7^2}{49^2 \times 7^5}$

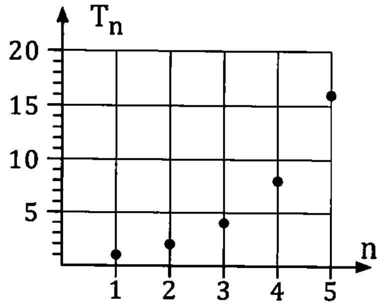
11. Given that $2^n \div 2^m = 2^{n-m}$ explain why it then makes sense for 2^0 to be equal to 1.

12. Copy and complete the following table.

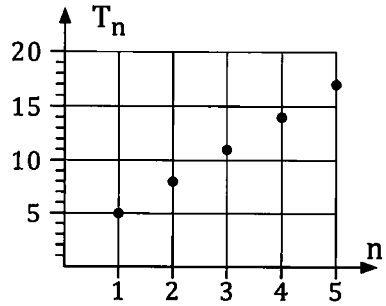
	$T_1, T_2, T_3, T_4, T_5, \ldots$	Recursively defined.
(a)		$T_n = T_{n-1} + 5,\ T_1 = 17$
(b)		$T_{n+1} = T_n - 7,\ T_1 = 100$
(c)		$T_n = 3T_{n-1},\ T_1 = 5$
(d)	6, 10, 14, 18, 22, ...	
(e)	2, 6, 18, 54, 162, ...	
(f)	17, 9, 1, -7, -15, ...	

13. Each of the two graphs shown below show the terms of a sequence. One sequence involves a recursive rule of the form $T_{n+1} = T_n + a$ and the other involves a recursive rule of the form $T_{n+1} = kT_n$ where a and k are constants.
Determine the value of a and k and hence determine T_{20} for each sequence.

(a)



(b)



14. (a) Without the assistance of a calculator, evaluate ${}^{6}C_4$.

    (b) Show that ${}^{n}C_1 = n$ and ${}^{n}C_2 = \dfrac{n(n-1)}{2}$.

15. Find the value of x and define each sequence recursively if the three terms

$$8,\ x,\ 50$$

are, in that order, the first three terms of (a) an arithmetic progression,
(b) a geometric progression.

16. The resale value of a particular item of machinery, t years after purchase, is expected to be \$$V$ where $V \approx 250\,000\,(0{\cdot}82)^t$.
Use your calculator to view the graph of V for $0 \leq t \leq 15$.
Using your graph, or by some other method, determine in how many years time the value will be (a) 50% of its current (i.e. $t = 0$) value.
(b) 25% of its current (i.e. $t = 0$) value.

17. Rosalyn does 30 minutes of dance practice each day. As the national championships approach Rosalyn decides to increase the amount she does by 3 minutes each day for the 20 days prior to the championships i.e. 20 days prior she will practise for 33 minutes, 19 days prior she will practise for 36 minutes etc.
Express the sequence 30, 33, 36, 39, 42, using recursive notation.
Under this scheme for how long will Rosalyn practise 1 day prior to the championships?
For how long does Rosalyn practise in total during these 20 days prior to the championships?

18. A competition advertises that the 1st prize is 1 million dollars. Whilst the winner of this prize will indeed receive \$1 million they will not receive it all at once. The prize conditions state that the winner will receive \$50000 immediately the win is announced, followed by \$50000 each year thereafter, on the anniversary of the first payment, for 19 further payments.
Compare the value of the following accounts after twenty years:
A: An account opened with a 1 million dollar investment, earning interest at 6% per annum compounded annually and left untouched for the 20 years.
B: An account opened with a \$50000 payment followed by a further \$50000 invested each year thereafter for 19 further payments, the account earning interest at 6% per annum compounded annually and, other than the regular annual injections of \$50000, left untouched for 20 years.
How much do the organisers have to have available "now" (i.e. at the time the winner is announced), rounded up to the next dollar, in order to meet their financial commitments to the winner, if they are to pay the initial \$50000 from this "available now" fund, and invest the rest in an account also paying a constant 6% interest per annum compounded annually, with the aim of being able to pay the nineteen annual amounts from this account, with the account balance reduced to zero at the end of this time?